Multiple Linear Regression
Categorical Predictors

January 20, 2015
Biostatistics 208

Overview

- Linear regression model for multiple predictors
- Categorical predictors in the linear model
  - binary predictors
  - multilevel categorical predictors
- *F* tests with categorical predictors
- Testing for linear trend with ordinal categorical predictors
- Interpretation of models with log-transformed outcomes & predictors

Definition of multi-predictor linear regression model

- Linear model for a continuous outcome Y and p predictors:

  - mean of Y is linearly related to the x's:

$$\mathrm{E}\left[y|x_1, x_2, \cdots, x_p\right] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p$$

  - individual values of Y are linearly related to the x's, with normally distributed errors:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \varepsilon$$

- β_0 is the mean of y when all x's are 0

- β_j is the change in the mean of y associated with a unit increase in x_j, holding the values of all the other x's fixed (except in complex models, e.g. with interactions)

- Coefficients estimated via least squares

- Coefficients for each predictor account for (or "control for") the presence of the other predictors

Multilevel Categorical Predictors

- Codes for each level of a multi-level classification
- Nominal example
  - race/ethnicity

```
. codebook raceth
-------------------------------------------
raceth                       race/ethnicity
-------------------------------------------

                  type:  numeric (byte)
                 label:  raceth

                 range:  [1,3]                          units:  1
         unique values:  3                          missing .:  0/2763

            tabulation:  Freq.   Numeric  Label
                          2451         1  White
                           218         2  African American
                            94         3  Other
```

Multilevel Categorical Predictors (continued)

- Entering a categorical predictor directly in a linear regression model does not give desired results:

```
. regress glucose raceth if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  1,  2030) =    4.20
       Model |  398.460556     1  398.460556           Prob > F      =  0.0406
    Residual |  192619.238  2030  94.8863243           R-squared     =  0.0021
-------------+------------------------------           Adj R-squared =  0.0016
       Total |  193017.699  2031  95.0357946           Root MSE      =   9.741

------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      raceth |   1.140344   .5564744     2.05   0.041      .049024    2.231665
       _cons |   95.39214   .6555531   145.51   0.000     94.10651    96.67776
------------------------------------------------------------------------------
```

- Coefficient for `raceth` gives the change in average glucose associated with a one-unit increase in `raceth`. This makes no sense!

Multilevel Categorical Predictors (continued)

- Want the regression model to estimate differences in average glucose between levels of race/ethnicity without assuming linearity

- Solution: Create separate binary indicators (dummy variables) for each level of `raceth`

```
. tabulate raceth, generate(re_cat)

  race/ethnicity |      Freq.     Percent        Cum.
-----------------+-----------------------------------
           White |      2,451       88.71       88.71
African American |        218        7.89       96.60
           Other |         94        3.40      100.00
-----------------+-----------------------------------
           Total |      2,763      100.00
```

```
. regress glucose re_cat* if diabetes==0
```
(*recall that we wish to exclude diabetic women in this analysis*)

```
note: re_cat2 omitted because of collinearity

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  2,  2029) =    2.67
       Model |  507.191904     2  253.595952           Prob > F      =  0.0693
    Residual |  192510.507  2029  94.8795007           R-squared     =  0.0026
-------------+------------------------------           Adj R-squared =  0.0016
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7406

------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     re_cat1 |  -.3783718   .9034809    -0.42   0.675    -2.150219    1.393475
     re_cat2 |          0  (omitted)
     re_cat3 |   2.756824   1.609274     1.71   0.087     -.399178    5.912826
       _cons |   96.93548   .8747327   110.82   0.000     95.22002    98.65095
------------------------------------------------------------------------------
```

Note: Stata only needs 2 indicator variables to estimate the differences in average outcome between the 4 race/eth. categories if the constant is included, so it drops one.

Multilevel Categorical Predictors (continued)

- Preceding categorical predictors with `i.` in regression formulae in Stata (versions 11+) automatically creates indicator variables necessary to represent multiple levels of a categorical predictor:

```
. regress glucose i.raceth if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  2,  2029) =    2.67
       Model |  507.191904     2  253.595952           Prob > F      =  0.0693
    Residual |  192510.507  2029  94.8795007           R-squared     =  0.0026
-------------+------------------------------           Adj R-squared =  0.0016
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7406

------------------------------------------------------------------------------
         glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------------+------------------------------------------------------------
          raceth |
African American |   .3783718   .9034809     0.42   0.675    -1.393475    2.150219
           Other |   3.135196   1.369572     2.29   0.022     .4492825    5.821109
                 |
           _cons |   96.55711   .2260983   427.06   0.000      96.1137    97.00052
------------------------------------------------------------------------------
```

Note: Stata chooses the lowest numerical level of the categorical variable as the reference category, and omits the indicator variable for this level this from the model

Multilevel Categorical Predictors (continued)

- Definition of indicator variables generated for `raceth` by `i.` with `regress` in Stata 11-13:

Category	`1.raceth`	`2.raceth`	`3.raceth`
White (1)	1	0	0
African American (2)	0	1	0
Other (3)	0	0	1

- Resulting regression model:

$$E[\texttt{glucose} \mid \texttt{raceth}] = \beta_0 + \beta_2 \texttt{2.raceth} + \beta_3 \texttt{3.raceth}$$

- With constant term in the model, don't need the indicator for the 1st category (taken as the baseline level)

- Coefficient for each indicator gives the difference in mean glucose between that level and the baseline level. For example:

Mean glucose for level 2: $\beta_0 + \beta_2$

Mean glucose for level 1: β_0

Difference in mean glucose between levels 2 &1: β_2

Difference in mean glucose between levels 3 &1: β_3

Multilevel Categorical Predictors (continued)

- Use of `lincom` to get mean glucose for all levels of race/ethnicity:

```
. quietly regress glucose i.raceth if diabetes==0

. lincom _cons                          white (1)
------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |   96.55711   .2260983   427.06   0.000      96.1137    97.00052
------------------------------------------------------------------------------


. lincom _cons + 2.raceth    african american (2)
-------------+----------------------------------------------------------------
         (1) |   96.93548   .8747327   110.82   0.000     95.22002    98.65095
------------------------------------------------------------------------------


. lincom _cons + 3.raceth    other (3)
-------------+----------------------------------------------------------------
         (1) |   99.69231    1.35078    73.80   0.000     97.04325    102.3414
------------------------------------------------------------------------------
```

- Check results:

```
. tab raceth if diabetes==0, sum(glucose)
race/ethnic | Summary of fasting glucose (mg/dl)
        ity |        Mean   Std. Dev.       Freq.
------------+------------------------------------
      White |   96.557112   9.6006499        1856
 African A  |   96.935484   10.933459         124
      Other |   99.692308   11.569972          52
------------+------------------------------------
```

Multilevel Categorical Predictors

- Ordinal example
  - grading of physical activity in HERS study (coded 1-5)

```
. codebook physact

-------------------------------------------------------------------
physact   comparative physical activity
-------------------------------------------------------------------
                  type:  numeric (byte)
                 label:  physact
                 range:  [1,5]                     units:  1
         unique values:  5                     missing .:  0/2763
           tabulation:  Freq.   Numeric  Label
                          197         1  much less active
                          503         2  somewhat less active
                          919         3  about as active
                          838         4  somewhat more active
                          306         5  much more active
```

Multilevel Categorical Predictors (continued)

- Regression model for dependence of `glucose` on an ordinal categorical measure:

```
. regress glucose physact if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  1,  2030) =   16.95
       Model |  1598.22653     1  1598.22653           Prob > F      =  0.0000
    Residual |  191419.472  2030  94.2953065           R-squared     =  0.0083
-------------+------------------------------           Adj R-squared =  0.0078
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7106

------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     physact |  -.8465276   .2056208    -4.12   0.000    -1.249777   -.4432778
       _cons |   99.46872   .7153373   139.05   0.000     98.06585    100.8716
------------------------------------------------------------------------------
```

- Model excludes individuals with diabetes
- Coefficient for `physact` gives the change in average glucose associated with a one-unit increase in `physact`
- `physact` is not a continuous measure, and we usually do not want to assume that glucose changes in the same way between all levels of activity

Multilevel Categorical Predictors (continued)

- Using `i.` syntax in `regress`:

```
. regress glucose i.physact if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  4,  2027) =    4.43
       Model |  1673.09022     4  418.272554           Prob > F      =  0.0014
    Residual |  191344.609  2027  94.3979322           R-squared     =  0.0087
-------------+------------------------------           Adj R-squared =  0.0067
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7159

------------------------------------------------------------------------------------------
               glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-----------------------+------------------------------------------------------------------
               physact |
somewhat less active   |  -.8584489   1.084152    -0.79   0.429    -2.984617    1.267719
      about as active  |  -1.226199   1.011079    -1.21   0.225     -3.20906    .7566629
somewhat more active   |  -2.433855   1.010772    -2.41   0.016    -4.416114    -.451595
    much more active   |  -3.277704   1.121079    -2.92   0.003    -5.476291   -1.079116
                       |
                 _cons |   98.42056   .9392676   104.78   0.000     96.57853    100.2626
------------------------------------------------------------------------------------------
```

Multilevel Categorical Predictors (continued)

- Using `b( )` syntax to specify baseline category:

```
. regress glucose ib(5).physact if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  4,  2027) =    4.43
       Model |  1673.09022     4  418.272554           Prob > F      =  0.0014
    Residual |  191344.609  2027  94.3979322           R-squared     =  0.0087
-------------+------------------------------           Adj R-squared =  0.0067
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7159

---------------------------------------------------------------------------------------
              glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
----------------------+----------------------------------------------------------------
              physact |
    much less active  |   3.277704   1.121079     2.92   0.003     1.079116    5.476291
somewhat less active  |   2.419255   .8171635     2.96   0.003     .8166866    4.021823
     about as active  |   2.051505    .717392     2.86   0.004     .6446024    3.458407
somewhat more active  |   .8438489   .7169593     1.18   0.239    -.562205     2.249903
                      |
                _cons |   95.14286   .6120416   155.45   0.000     93.94256    96.34315
---------------------------------------------------------------------------------------
```

Note: Compare results to slide #12

Multilevel Categorical Predictors (continued)

- Definition of indicator variables generated for `physact` by `i.` with `regress`:

Physical activity category	`1.physact`	`2.physact`	`3.physact`	`4.physact`	`5.physact`
much less active (1)	1	0	0	0	0
somewhat less active (2)	0	1	0	0	0
about as active (3)	0	0	1	0	0
somewhat more active (4)	0	0	0	1	0
much more active (5)	0	0	0	0	1

- Resulting regression model:

$$E[\text{glucose} \mid \texttt{physact}] = \beta_0 + \beta_2 \texttt{2.physact} + \beta_3 \texttt{3.physact} + \beta_4 \texttt{4.physact} + \beta_5 \texttt{5.physact}$$

- With constant term in the model, don't need the indicator for the 1st category (taken as the baseline level)

- Coefficient for each indicator gives the difference in mean glucose between that level and the baseline level. For example:

Mean glucose for level 2: $\beta_0 + \beta_2$

Mean glucose for level 1: β_0

Difference in mean glucose between levels 2 &1: β_2

Multilevel Categorical Predictors (continued)

- Use of `lincom` to get mean glucose for all levels of physical activity:

```
. quietly regress glucose i.physact if diabetes==0

. lincom _cons                          much less active (1)
------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |   98.42056   .9392676   104.78   0.000     96.57853    100.2626
------------------------------------------------------------------------------
. lincom _cons + 2.physact              somewhat less active (2)
-------------+----------------------------------------------------------------
         (1) |   97.56211   .5414437   180.19   0.000     96.50027    98.62396
------------------------------------------------------------------------------
. lincom _cons + 3.physact              about as active (3)
-------------+----------------------------------------------------------------
         (1) |   97.19436   .3742409   259.71   0.000     96.46043     97.9283
------------------------------------------------------------------------------
. lincom _cons + 4.physact              somewhat more active (4)
-------------+----------------------------------------------------------------
         (1) |   95.98671   .3734108   257.05   0.000      95.2544    96.71902
------------------------------------------------------------------------------
. lincom _cons + 5.physact              much more active (5)
-------------+----------------------------------------------------------------
         (1) |   95.14286   .6120416   155.45   0.000     93.94256    96.34315
------------------------------------------------------------------------------
```

Multilevel Categorical Predictors (continued)

- The linear model including binary indicators to represent levels of a categorical predictor allows comparisons between any desired levels of physical activity

- Use of `lincom` to get difference in mean glucose between two specified levels of physical activity (e.g. level 5 compared to level 3):

Mean glucose for level 5: $\beta_0 + \beta_5$

Mean glucose for level 3: $\beta_0 + \beta_3$

Difference: $\beta_5 - \beta_3$

```
. lincom 5.physact - 3.physact

 ( 1)  - 3.physact + 5.physact = 0

------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |  -2.051505    .717392    -2.86   0.004    -3.458407   -.6446024
------------------------------------------------------------------------------
```

Multilevel Categorical Predictors (continued)

- Use of `contrast` to get difference in mean glucose between two specified levels of physical activity (e.g. level 5 compared to level 3):

```
. contrast {physact 0 0 -1 0 1}, effects

Contrasts of marginal linear predictions

Margins      : asbalanced

------------------------------------------------
             |         df           F        P>F
-------------+----------------------------------
     physact |          1        8.18     0.0043
             |
    Residual |       2027
------------------------------------------------

------------------------------------------------------------------------------
             |   Contrast   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     physact |
        (1)  |  -2.051505    .717392    -2.86   0.004    -3.458407   -.6446024
------------------------------------------------------------------------------
```

Multilevel Categorical Predictors (continued)

- Note: in Stata 10 (and earlier versions) the `i.` syntax works only in conjunction with the `xi:` command (both types of commands work in Stata 11+)

```
. xi:regress glucose i.physact if diabetes==0
i.physact         _Iphysact_1-5       (naturally coded; _Iphysact_1 omitted)

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  4,  2027) =    4.43
       Model |  1673.09022     4  418.272554           Prob > F      =  0.0014
    Residual |  191344.609  2027  94.3979322           R-squared     =  0.0087
-------------+------------------------------           Adj R-squared =  0.0067
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7159

------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
 _Iphysact_2 |  -.8584489   1.084152    -0.79   0.429    -2.984617    1.267719
 _Iphysact_3 |  -1.226199   1.011079    -1.21   0.225     -3.20906    .7566629
 _Iphysact_4 |  -2.433855   1.010772    -2.41   0.016    -4.416114    -.451595
 _Iphysact_5 |  -3.277704   1.121079    -2.92   0.003    -5.476291   -1.079116
       _cons |   98.42056   .9392676   104.78   0.000     96.57853    100.2626
------------------------------------------------------------------------------
```

Note: Results identical to the models on previous slides.

Multilevel Categorical Predictors (continued)

- Use of `lincom` to get mean glucose for all levels of physical activity using Stata 10 syntax:

```
. lincom _cons                          much less active (1)
------------------------------------------------------------------------------
     glucose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         (1) |   98.42056   .9392676   104.78   0.000     96.57853    100.2626
------------------------------------------------------------------------------
. lincom _cons + _Iphysact_2   somewhat less active (2)
-------------+----------------------------------------------------------------
         (1) |   97.56211   .5414437   180.19   0.000     96.50027    98.62396
------------------------------------------------------------------------------
. lincom _cons + _Iphysact_3   about as active (3)
-------------+----------------------------------------------------------------
         (1) |   97.19436   .3742409   259.71   0.000     96.46043     97.9283
------------------------------------------------------------------------------
. lincom _cons + _Iphysact_4   somewhat more active (4)
-------------+----------------------------------------------------------------
         (1) |   95.98671   .3734108   257.05   0.000      95.2544    96.71902
------------------------------------------------------------------------------
. lincom _cons + _Iphysact_5   much more active (5)
-------------+----------------------------------------------------------------
         (1) |   95.14286   .6120416   155.45   0.000     93.94256    96.34315
------------------------------------------------------------------------------
```

Multilevel Categorical Predictors (continued)

- Testing overall impact of physical activity as a predictor of average glucose level in HERS
- *F* statistic in the standard output tests the *homogeneity* hypothesis that the true values for all of the (4) coefficients used to represent physical activity are equal to zero

```
. regress glucose i.physact if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  4,  2027) =    4.43
       Model |  1673.09022     4  418.272554           Prob > F      =  0.0014
    Residual |  191344.609  2027  94.3979322           R-squared     =  0.0087
-------------+------------------------------           Adj R-squared =  0.0067
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7159
------------------------------------------------------------------------------
Rest of output omitted ...
```

- The `testparm` command can be used to evaluate this, even when additional predictors are included in the model:

```
. testparm i.physact

 ( 1)  2.physact = 0
 ( 2)  3.physact = 0
 ( 3)  4.physact = 0
 ( 4)  5.physact = 0

       F(  4,  2027) =    4.43
            Prob > F =    0.0014
```

Multilevel Categorical Predictors (continued)

- The test performed by `testparm` on the previous slide is an example of using the *F* test to compare *nested* models.

1. Fit the *full model* including all the variable(s) of interest. Call this model 1 (fitted on previous slide).

2. Fit the *nested model* excluding the variable(s) of interest. Call this model 2:

```
. regress glucose if diabetes==0

      Source |       SS       df       MS              Number of obs =    2032
-------------+------------------------------           F(  0,  2762) =    0.00
       Model |           0     0           .           Prob > F      =       .
    Residual |  193017.699  2031  95.0357946           R-squared     =  0.0000
-------------+------------------------------           Adj R-squared =  0.0000
       Total |  193017.699  2031  95.0357946           Root MSE      =  9.7486
Rest of output omitted ...
```

3. Compute the following statistic comparing the residual sums of squares (RSS) and corresponding degrees of freedom between the two models:

$F = [(RSS_2 - RSS_1)/ (df_2 - df_1)] / (RSS_1 / df_1)$

(this statistic has the *F* distribution with ($df_2 - df_1$, df_2) degrees of freedom

```
. dis ((193017.699 - 191344.609)/(2031-2027)) / (191344.609/2027)
4.4309498 (F-statistic)
. dis 1-F(4,2031,4.4309498)
.00144076  (p-value)
```

4. *Note*: Don't do this by hand in practice. It can be more easily accomplished using the `testparm` command as shown above

Multilevel Categorical Predictors (continued)

- *F* test allows an overall test of the contribution of `physact` to the model by evaluating evidence for heterogeneity in group-specific means

- When only a single categorical predictor is included, results are identical to one-way ANOVA:

```
. anova glucose physact if diabetes==0
                          Number of obs =    2032     R-squared     =  0.0087
                          Root MSE      = 9.71586     Adj R-squared =  0.0067
                 Source | Partial SS    df       MS           F     Prob > F
            -----------+----------------------------------------------------
                  Model |  1673.09022     4  418.272554       4.43     0.0014
                        |
                physact |  1673.09022     4  418.272554       4.43     0.0014
                        |
               Residual |  191344.609  2027  94.3979322
            -----------+----------------------------------------------------
                  Total |  193017.699  2031  95.0357946
```

- individual *t-tests* only allow evaluation of each activity level to the baseline level
  - multiple tests harder to interpret, and can multiple testing inflate experiment-wise error rate (*EER*, see section 4.3.4 of textbook)

Multilevel Categorical Predictors (continued)

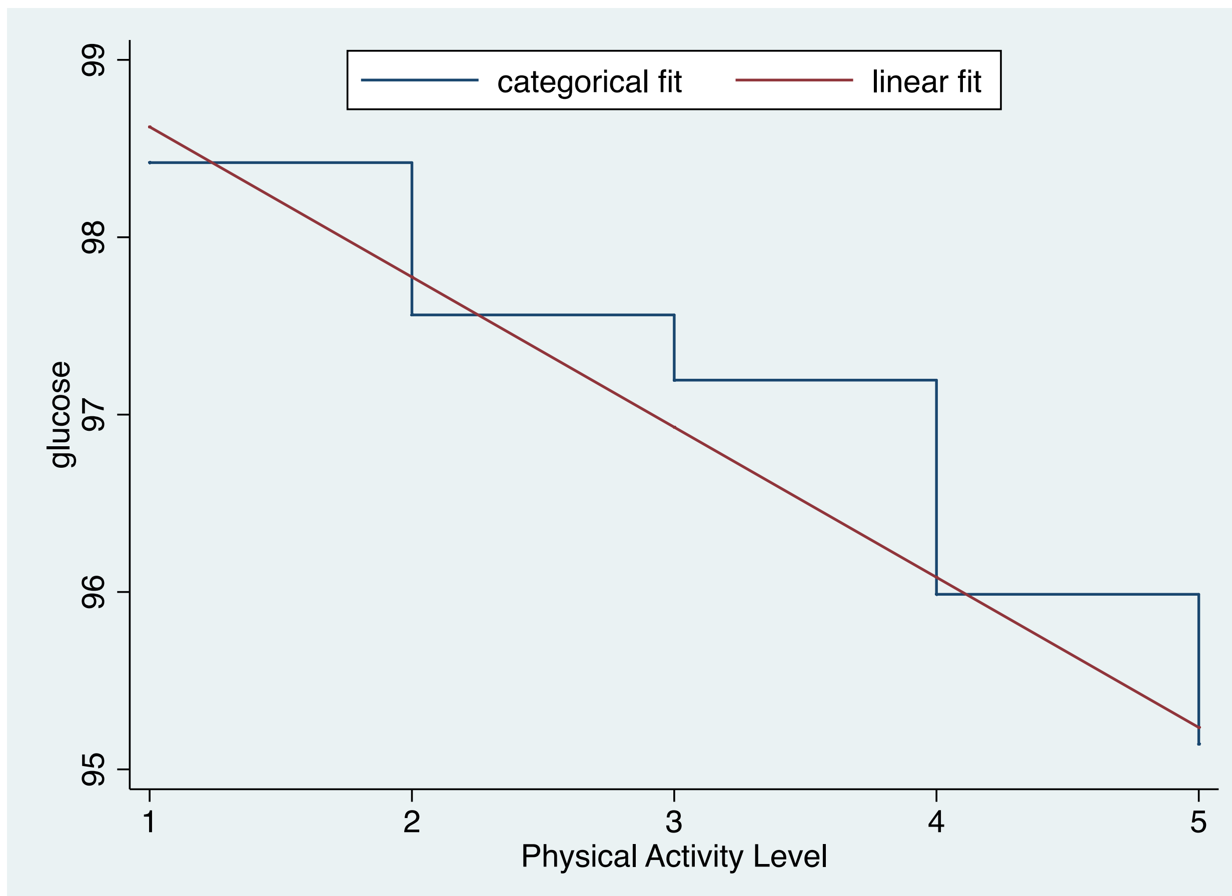
- *P*-values for individual mean comparisons can be corrected for multiple testing using the `contrast` statements in Stata

```
. contrast physact, mcompare(bonferroni) effects

Contrasts of marginal linear predictions

Margins      : asbalanced

------------------------------------------------
             |         df           F        P>F
-------------+----------------------------------
     physact |          4        4.43     0.0014
             |
    Residual |       2027
------------------------------------------------
Note: Bonferroni-adjusted p-values are reported
      for tests on individual contrasts only.

---------------------------
             |    Number of
             |  Comparisons
-------------+-------------
     physact |            4
---------------------------

------------------------------------------------------------------------------
             |                          Bonferroni            Bonferroni
             |   Contrast   Std. Err.       t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     physact |
(2 vs base)  |  -.8584489   1.084152    -0.79   1.000     -3.56876    1.851862
(3 vs base)  |  -1.226199   1.011079    -1.21   0.901    -3.753832    1.301434
(4 vs base)  |  -2.433855   1.010772    -2.41   0.065     -4.96072     .093011
(5 vs base)  |  -3.277704   1.121079    -2.92   0.014    -6.080331   -.4750759
------------------------------------------------------------------------------
```

Note: Compare *P*-values and 95% CIs to default model output

Trend Test: example

- Average glucose varies across levels of `physact`
- Lack of trend: best fitting line to mean outcome levels across levels is flat
- A significant test for trend indicates that the above line has non-zero slope



- Problem with trend testing based on fitting a line: labeling of categories is often arbitrary

Trend Test: example

- Evidence for the presence of linear trend can most easily be assessed using the `contrast` command:

```
. contrast q(1).physact, noeffects

Contrasts of marginal linear predictions

Margins      : asbalanced

------------------------------------------------
             |         df           F        P>F
-------------+----------------------------------
     physact |          1       12.11     0.0005
             |
    Residual |       2027
------------------------------------------------
```

- `q(1)` option requests linear trend
- based on *contrasts* of the model coefficients
- works for categories with *equal spacings* (see section 4.3.5 for alternatives)
- The null hypothesis is that the slope of the linear trend is zero - a significant *p*-value indicates evidence for linear trend

Trend Test: example

- Contrasts are explained in section 4.3.5 of text book
- Can be evaluated explicitly using the `contrast` command

```
.contrast {physact -2 -1 0 1 2}, noeffects

Contrasts of marginal linear predictions

Margins      : asbalanced
------------------------------------------------
             |         df           F        P>F
-------------+----------------------------------
     physact |          1       12.11     0.0005
             |
 Denominator |       2027
------------------------------------------------
```

Number of levels	Model Coefficients	Linear contrast in coefficients
3	β_2, β_3	$\beta_3 = 0$
4	$\beta_2, \beta_3, \beta_4$	$-\beta_2 + \beta_3 + 3\beta_4 = 0$
5	$\beta_2, \beta_3, \beta_4, \beta_5$	$-\beta_2 + \beta_4 + 2\beta_5 = 0$
6	$\beta_2, \beta_3, \beta_4, \beta_5, \beta_6$	$-3\beta_2 - \beta_3 + \beta_4 + 3\beta_5 + 5\beta_6 = 0$

See Table 4.8 in section 4.3.5 of textbook

Trend Test: example

- The following `test` command applied after the model including `physact` is fitted evaluates the significance of linear trend using the contrast coefficients explicitly:

```
. quietly regress glucose i.physact if diabetes==0
. test -2.physact + 4.physact + 2*5.physact=0

 ( 1)  - 2.physact + 4.physact + 2*5.physact = 0

       F(  1,  2027) =   12.11
            Prob > F =    0.0005
```

- The contrast is based on the table in the previous slide for a 5-level predictor
- Conclude that there is evidence for a linear trend in mean glucose across levels of `physact`
- Note that a linear trend may not be sufficient to characterize the change in average glucose across levels of `physact` - a departure from this trend may also be present
- Test for a departure from a linear trend is also possible

Trend Test: evaluating evidence for departure from linear trend

- Trend in levels of an ordinal categorical predictor can be characterized by presence of a linear component as well as a nonlinear part

- Testing for *departure from linear trend* allows evaluation of this (after presence of a trend is established)

- Can also use contrast command (see 3.7.5 for details):

```
. contrast q(2/4).physact, noeffects

Contrasts of marginal linear predictions

Margins      : asbalanced

------------------------------------------------
             |         df           F        P>F
-------------+----------------------------------
     physact |
 (quadratic) |          1        0.11     0.7411
     (cubic) |          1        0.01     0.9415
   (quartic) |          1        0.49     0.4859
       Joint |          3        0.26     0.8511
             |
 Denominator |       2027
------------------------------------------------
```

- `q(2/4)` option requests additional tests based on evaluating quadratic, cubic and quartic trend components (omitting linear component)

Results

- We see that there is a pattern of decrease in glucose with physical activity
- *p*-value for trend test is highly significant: there is a significant trend
- *Note* : evaluating trend by including a categorical predictor as a continuous score in a regression model, and using the associated *t*-test is generally not equivalent (and not preferred) to the general approach outlined in the past few slides

Multilevel Categorical Predictors (continued)

Homogeneity Between Outcome Means for Levels of a Categorical Predictor vs. Trend in Mean over Levels:

- Homogeneity: no difference in mean outcome between the groups coded by the categorical predictor

- Trend: difference between groups which implies a trend in mean glucose with increasing levels of the categorical predictor

- "Trend" only makes sense if predictor variable is ordered is some way

- Trend test can be based on a specified combination of coefficients (a so-called *contrast*) and tested using `test` or `lincom` commands in Stata (section 4.3.5 of textbook)

Interpreting results for log-transformed variables

- Positive continuous variables commonly log-transformed in regression models
  - *outcomes*: normalize and equalize variance
  - *predictors*: address non-linearity or interaction issues, reduce influence of outliers
- Both natural log (ln) and $\log_{10}$ (log10) transformations frequently used
- How does this affect interpretation of regression coefficients?

Brief review of exponentials and natural logarithms

Properties of exponentials

$e = 2.7182818...$

$e^a = \exp(a)$

$e^a e^b = e^{a+b}$, $e^a / e^b = e^{a-b}$

$(e^a)^b = e^{ab}$

$e^0 = 1$

$-\infty < a \leq 0$, then $0 < e^a \leq 1$

$0 < a < \infty$, then $1 < e^a < \infty$

Properties of logarithms

$\log(e) = 1$

$\log(e^a) = a$, $\exp(\log(a)) = a$

$\log(ab) = \log(a) + \log(b)$, $\log(a/b) = \log(a) - \log(b)$

$\log(a^b) = b \log(a)$

$\log(1) = 0$

$0 < a \leq 1$, then $-\infty < \log(a) \leq 0$

$1 < a < \infty$, then $0 < \log(a) < \infty$

Similar properties apply to logs of other bases (e.g. $\log_{10}(10^a) = a$, $10^{(\log10(a))} = a$)

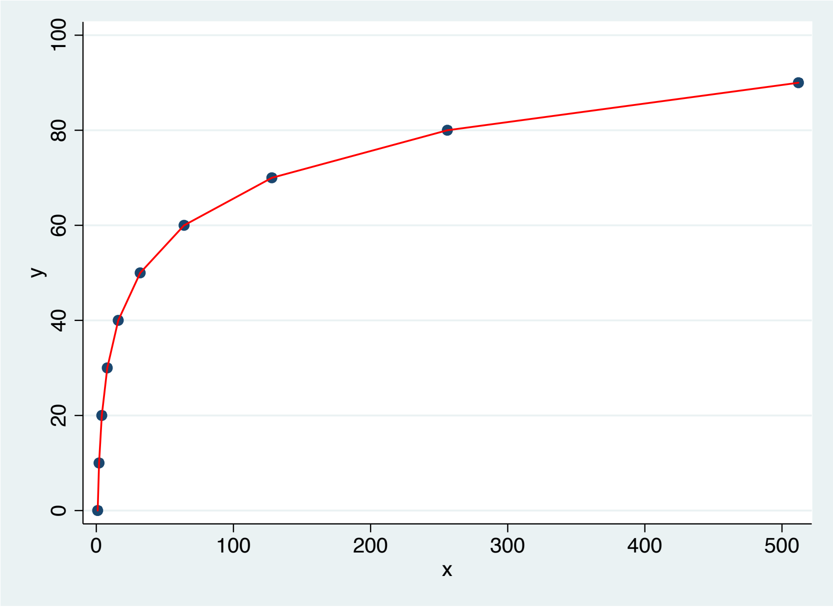
Log-transformed predictors

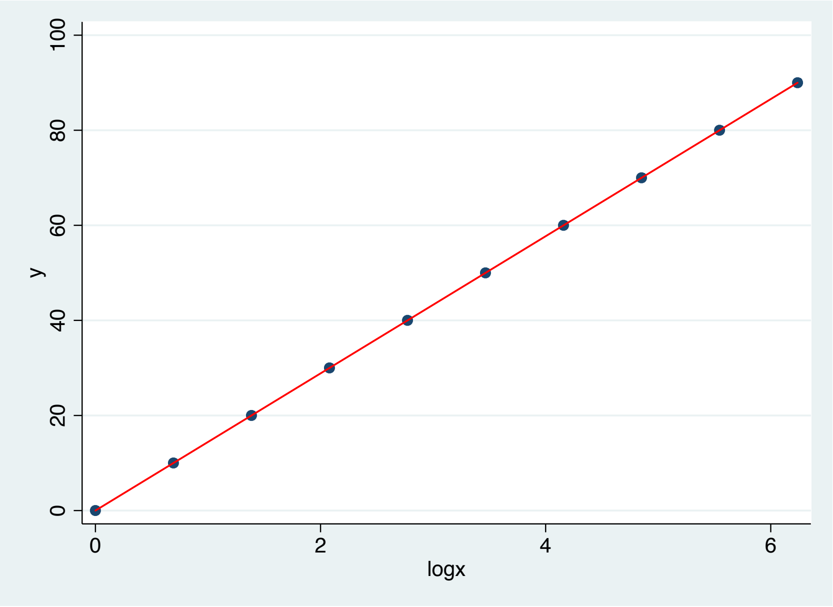
- We'll assume that the natural logarithm (i.e. log to the base e) of a variable x is denoted $\log(x)$, or equivalently, $\ln(x)$
  - log to another base (e.g. 10) will be denoted $\log_{10}(x)$
- Example: models for diminishing response (y) with increasing dose (x):

$$\mathrm{E}\left[y|x\right] = \beta_0 + \beta_1 \log(x)$$

  - For example, $\beta_0 = 0$, $\beta_1 = 10/\log(2)$ models a 10-unit increase in $\mathrm{E}[y|x]$ for each doubling of x

x	$\mathrm{E}[y\|x]$
1	0
2	10
4	20
8	30
16	40





Log-transformed predictors

- In a linear regression model with a natural log-transformed predictor x (could be one of several included predictors), β_1 gives the increase in $E[y|x]$ for a one-unit increase in $\log(x)$
  - This corresponds to an e-fold (e=2.718..) increase in x
- If predictor x is $\log_{10}$-transformed, β_1 gives the increase in $E[y|x]$ for a one-unit increase in $\log_{10}(x)$
  - This corresponds to an 10-fold increase in x
- How can we get more interpretable estimates?

Log-transformed predictors

- For a natural log-transformed predictor x

$$\mathrm{E}\left[y|x\right] = \beta_0 + \beta_1 \log(x)$$

  - $\beta_1 \log(1 + k/100)$: gives change in $\mathrm{E}[y|x]$ for a $k\%$ increase in x:

$$\left[\beta_0 + \beta_1 \log\left[x(1 + k/100)\right]\right] - \left[\beta_0 + \beta_1 \log(x)\right] = \beta_1 \log\left(\frac{x(1 + k/100)}{x}\right) = \beta_1 \log\left(1 + k/100\right)$$

- For a $\log_{10}$-transformed predictor x

$$\mathrm{E}\left[y|x\right] = \beta_0 + \beta_1 \log_{10}(x)$$

  - $\beta_1 \log_{10}(1 + k/100)$: gives change in $\mathrm{E}[y|x]$ for a $k\%$ increase in x

- *Note:*

  - Same interpretation holds for models with multiple predictors

  - Use `nlcom` to get estimates with confidence intervals (`nlcom` is the analog of `lincom` appropriate for estimates and CIs of nonlinear functions of coefficients)

Example: SBP and log-transformed creatinine in HERS

```
. regress sbp lncreat age diabetes

      Source |       SS       df       MS              Number of obs =    2761
-------------+------------------------------           F(  3,  2757) =   61.70
       Model |  62724.9665     3  20908.3222           Prob > F      =  0.0000
    Residual |  934341.036  2757  338.897728           R-squared     =  0.0629
-------------+------------------------------           Adj R-squared =  0.0619
       Total |  997066.003  2760  361.255798           Root MSE      =  18.409

------------------------------------------------------------------------------
         sbp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
     lncreat |   9.211717   1.682269     5.48   0.000     5.913081    12.51035
         age |    .444083   .0536661     8.27   0.000     .3388531    .5493129
    diabetes |   6.426345   .8007529     8.03   0.000     4.856209    7.996481
       _cons |   103.2765   3.600996    28.68   0.000     96.21558    110.3374
------------------------------------------------------------------------------

. nlcom _b[lncreat]*log(1 + 0.5)

       _nl_1:  _b[lncreat]*log(1 + 0.5)

------------------------------------------------------------------------------
         sbp |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _nl_1 |    3.73503   .6821016     5.48   0.000     2.398135    5.071924
------------------------------------------------------------------------------
```

Interpretation: average SBP increases 3.7 mmHg with each 50% increase in creatinine

Natural log-transformed outcomes

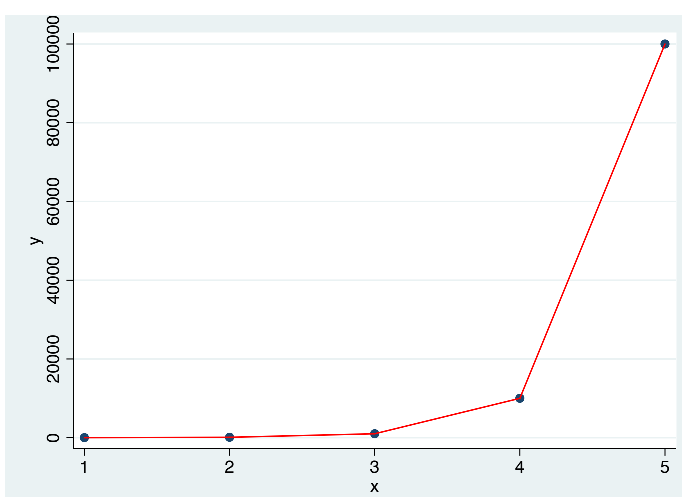
- Models increasing or decreasing exponential dose-response

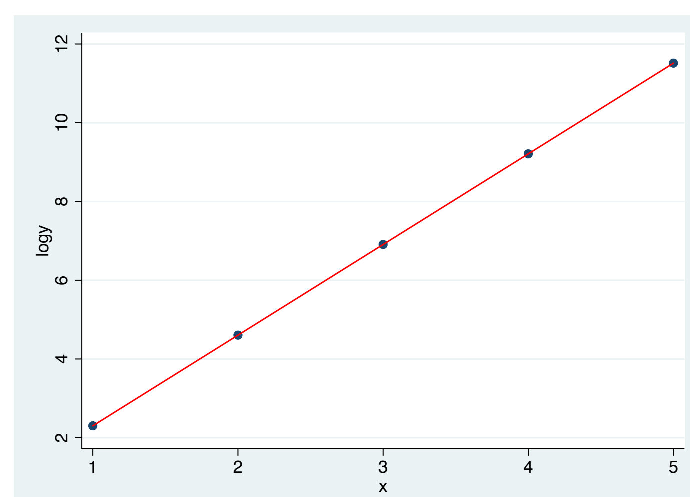
$$\mathrm{E}\left[\log(y)|x\right] = \beta_0 + \beta_1 x$$

  - For example, $\beta_0 = 0$, $\beta_1 = \pm \log(10)$ models a 10-fold (relative) change in $\mathrm{E}[y|x]$ for each unit increase in x

	E[y|x]	
x	$\beta_1 = \log(10)$	$\beta_1 = -\log(10)$
1	10	0.1
2	100	0.01
3	1000	0.001
4	10000	0.0001
5	100000	0.00001

For $\beta_1 = \log(10)$:





- Interpretation is approximate: $\mathrm{E}[\log(y)] \neq \log(\mathrm{E}[y])$

Natural log-transformed outcomes

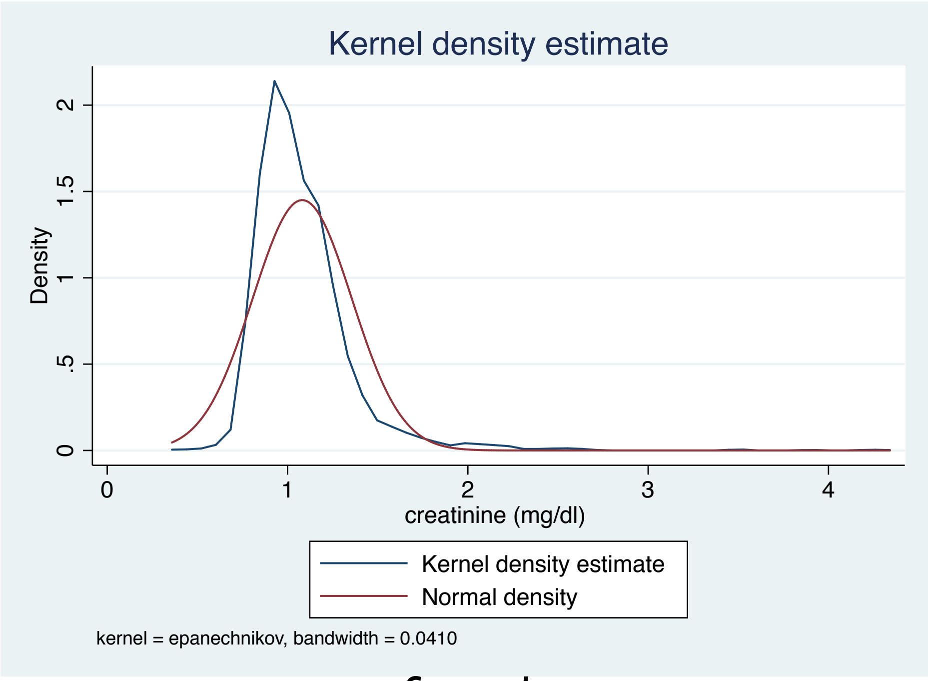
- In a linear regression model with a natural log-transformed outcome y,
  - β_1 gives the *additive* increase in $E[\log(y)|x]$ for a one-unit increase in x
  - $\exp(\beta_1) = e^{\beta 1}$ gives the *relative* increase in $E[y|x]$ for a one-unit increase in x (e.g. if $e^{\beta 1} = 1.5$, a 1.5-fold increase in $E[y|x]$ for a 1-unit increase in x from 1 to 1.5, 10 to 15, 100 to 150, …)
- The *percent increase* in $E[y|x]$ for a unit increase in x is given by $100(e^{\beta 1} - 1)$
  - i.e. $e^{\beta 1} = 1.5$ represents a 50% increase in $E[y|x]$ for each 1-unit increase in x
- Adding a small amount to outcomes makes it possible to apply log transformation to a variable that is zero for some observations (e.g. $\log(y+1)$)

Natural log-transformed outcomes

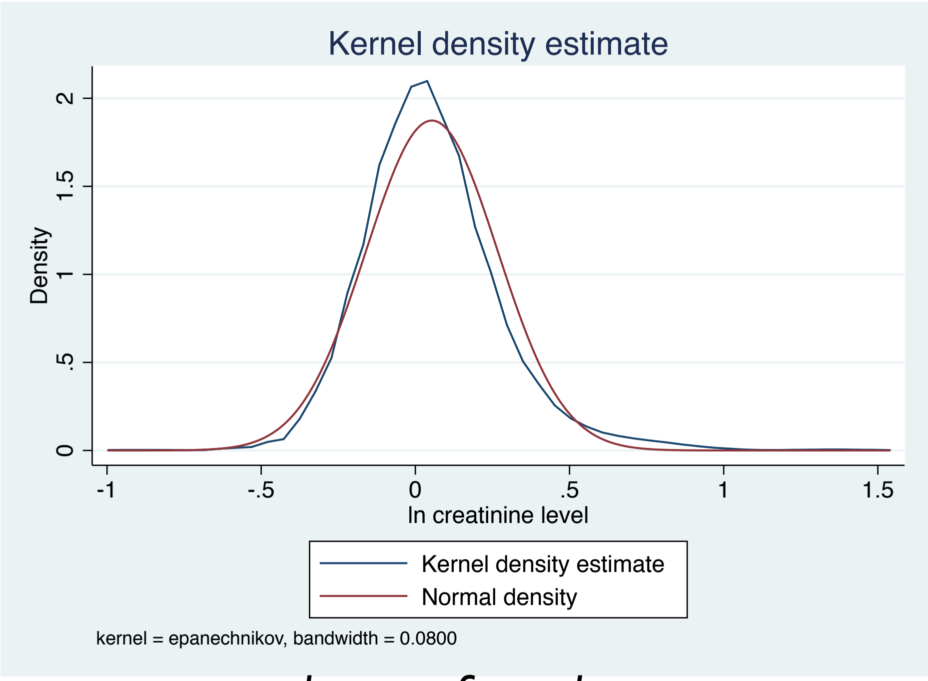
- For untransformed predictors:
  - use option `eform("exp(beta)")` to have `regress` display relative effect $e^{\beta 1}$
  - use `lincom` with `eform` option to get relative increase in (untransformed) outcome per *k*-unit increase in predictor (where *k* is an arbitrary integer)
  - use `nlcom` to get percent increase in outcome per *k*-unit increase in

Example: effect of age on log-transformed creatinine in HERS

```
. regress lncreat bmi age, eform("exp(beta)")

      Source |       SS       df       MS              Number of obs =    2756
-------------+------------------------------           F(  2,  2753) =   55.00
       Model |  4.76569402     2  2.38284701           Prob > F      =  0.0000
    Residual |  119.265116  2753  .043321873           R-squared     =  0.0384
-------------+------------------------------           Adj R-squared =  0.0377
       Total |   124.03081  2755  .045020258           Root MSE      =  .20814

------------------------------------------------------------------------------
     lncreat |  exp(beta)   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         bmi |   1.003308   .0007303     4.54   0.000     1.001877    1.004741
         age |   1.006093   .0006076    10.06   0.000     1.004902    1.007285
       _cons |   .6405172   .0309547    -9.22   0.000     .5826076     .704183
-------------+----------------------------------------------------------------
```



untransformed



log-transformed

Example: effect of age on log-transformed creatinine in HERS

```
* relative increment in creatinine associated with 10-year increase in age
. lincom age*10, eform

 ( 1)  10*age = 0

------------------------------------------------------------------------------
      lncreat |     exp(b)   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          (1) |   1.062624   .0064171    10.06   0.000     1.050115    1.075282
-------------+----------------------------------------------------------------
```

- Creatinine increases 1.063-fold for each 10-year increase in age

```
. nlcom 100*(exp(_b[age]*10)-1)

       _nl_1:  100*(exp(_b[age]*10)-1)

------------------------------------------------------------------------------
      lncreat |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        _nl_1 |   6.262402   .6417065     9.76   0.000      5.00468    7.520123
------------------------------------------------------------------------------
```

- Creatinine increases 6.3% for each 10-year increase in age

Additive versus multiplicative models

- *Additive models* express the effect of a unit increase in a predictor (e.g. from x to $x+1$) on the average outcome as *additive*:

$$\mathrm{E}\left[y|x+1\right] = \mathrm{E}\left[y|x\right] + \beta$$

  - β gives *additive* change (increase/decrease) associated with a unit increase in x

- *Multiplicative models* express the effect of a unit increase in a predictor (e.g. from x to $x+1$) on the average outcome as *multiplicative*:

$$\mathrm{E}\left[y|x+1\right] = \mathrm{E}\left[y|x\right] \times \beta$$

  - β gives *relative* change associated with a unit increase in x
  - linear models for log-transformed outcomes are multiplicative with respect to the untransformed outcomes.

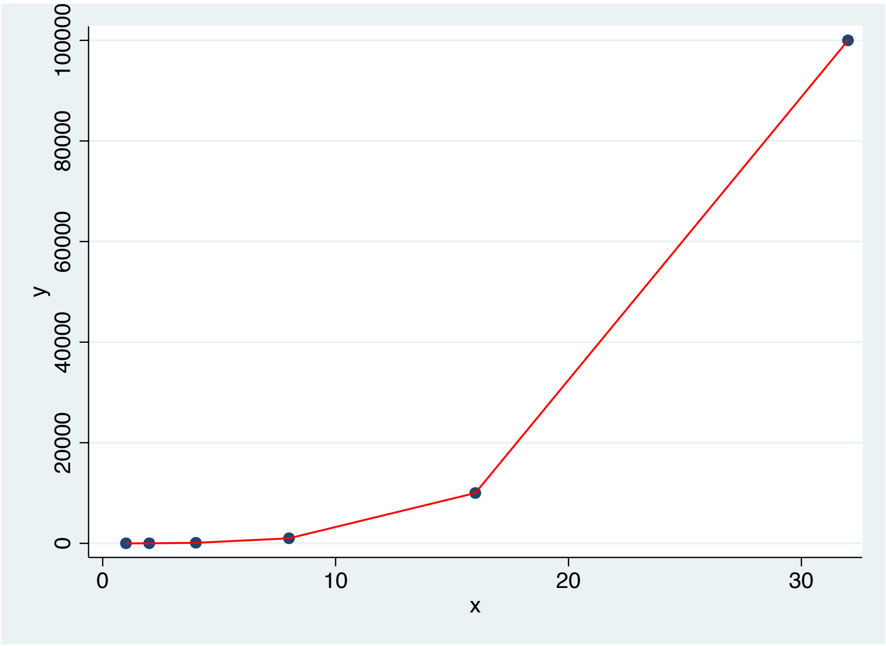
Outcome & predictor both log-transformed

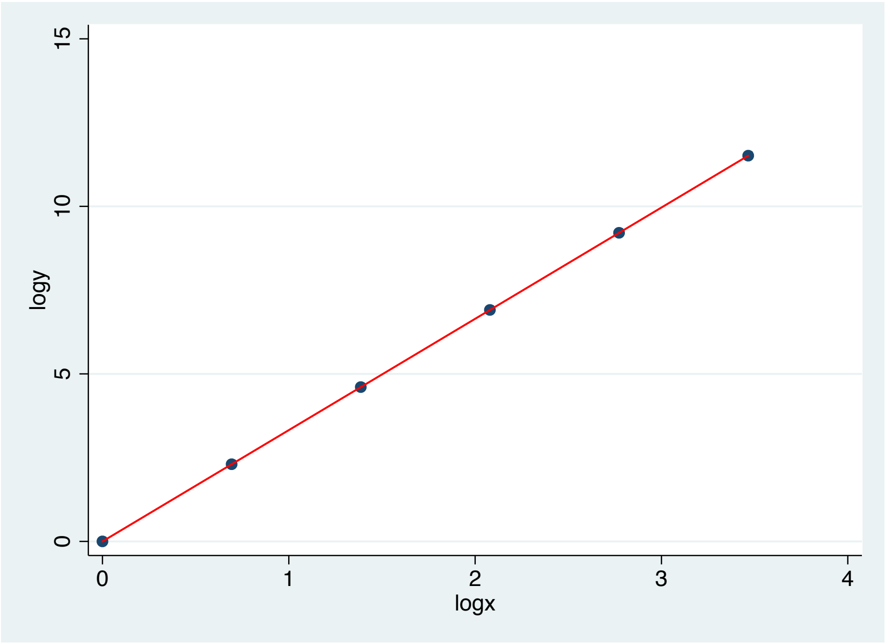
- Models exponential response to proportional increases in dose

$$E\left[\log(y)|x\right] = \beta_0 + \beta_1 \log(x)$$

  - For example, $\beta_1 = \log(10)/\log(2)$ (with $\beta_0 = 0$) models a 10-fold (relative) change in $E[y|x]$ for each doubling of x

x	$E[y\|x]$
1	1
2	10
4	100
8	1000
16	10000





Outcome & predictor both log-transformed

- In a linear regression model with a natural log-transformed predictor x and outcome y:
  - $\exp(\beta_1 \log(1 + k/100))$ gives the *relative* increase in $E[y|x]$ for a *k*% increase in x
  - $100 (\exp(\beta_1 \log(1 + k/100)) - 1)$ gives the *percent* increase in $E[y|x]$ for a *k*% increase in x
- use `eform` with `regress`, `nlcom` for estimates with CIs
- See VGSM, Sect. 4.7.5 for more details - examples in lab this week

Example: Log-transformed creatinine and TG in HERS

```
. regress lncreat bmi age lntgl, eform("exp(beta)")

      Source |       SS       df       MS              Number of obs =    2752
-------------+------------------------------           F(  3,  2748) =   42.15
       Model |  5.45380764     3  1.81793588           Prob > F      =  0.0000
    Residual |  118.530536  2748  .043133383           R-squared     =  0.0440
-------------+------------------------------           Adj R-squared =  0.0429
       Total |  123.984344  2751  .045068827           Root MSE      =  .20769

------------------------------------------------------------------------------
     lncreat |  exp(beta)   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         bmi |   1.002752   .0007412     3.72   0.000       1.0013    1.004206
         age |   1.006025   .0006069     9.96   0.000     1.004835    1.007216
       lntgl |   1.041705    .010456     4.07   0.000     1.021403     1.06241
       _cons |   .5321261   .0353937    -9.48   0.000     .4670604     .606256
------------------------------------------------------------------------------

. nlcom 100*(exp(_b[lntgl]*log(1.25))-1)

       _nl_1:  100*(exp(_b[lntgl]*log(1.25))-1)

------------------------------------------------------------------------------
     lncreat |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
       _nl_1 |   .9159048   .2260289     4.05   0.000     .4728963    1.358913
```

Creatinine increases 0.9% for each 25% increase in TG

Summary

- Multiple linear regression model extends simple linear model and involves similar assumptions
- Categorical predictors factored into binary indicators (use `i.` with `regress` in Stata)
- Mean outcome for successive levels compared to a reference level
- Can perform: overall *F* test, trend test or examine pairwise comparisons
- Trend tests useful with ordinal categorical variables
- Interpretation of coefficients from models involving log-transformed outcomes/predictors