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Lecture 5

$$E[\cancel{g}h(x)] = \sum_{\cancel{x}\, i} h\cancel{g}(i)\, P(X=i)$$

$$E[u^X] = \sum_i u^i P(X=i) = g(u) \quad 0 \le u \le 1$$

Generating Function - Useful Facts

$$g(u) = P(X=0) + uP(X=1) + u^2P(X=2) + u^3P(X=3)$$

$$\begin{cases} g(0) = P(X=0) \\ g(1) = 1 \end{cases}$$

$$g'(u) = P(X=1) + 2u\,P(X=2) + 3u^2 P(X=3)$$

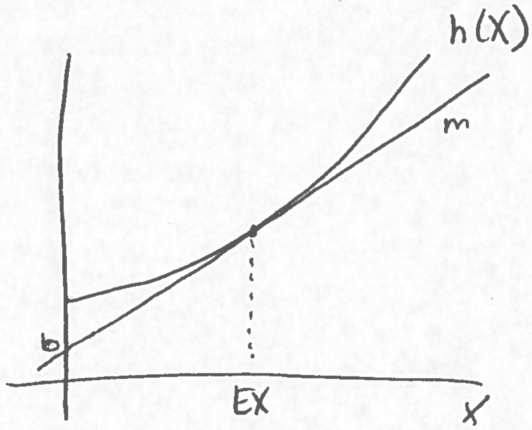
$$\begin{cases} g'(0) = P(X=1) \\ g'(1) = E[X] \end{cases}$$

$$g''(u) = 2P(X=2) + 3\cdot 2u\,P(X=3) + 4\cdot 3u^2 P(X=4)$$

$g''(u) \ge 0$; typically positive
(as long as $P(X \ge 2) \neq 0$)

$\Rightarrow$ convex (if you connect 2 pts the line joining them is always above the curve ;
the tangent lies below the curve)

①



$$h(X) \geq m(X - EX) + \cancel{b}\, h(EX)$$

$$E[h(X)] \geq E[m(X-EX) + h(EX)]$$

$$\geq mEX - mEX + E[h(EX)]$$

$$\boxed{E[h(x)] \geq h(EX)}$$ useful! Jensen's Inequality

eg. $E[X^2] \geq (EX)^2$

eg $h(x) = -\log x$

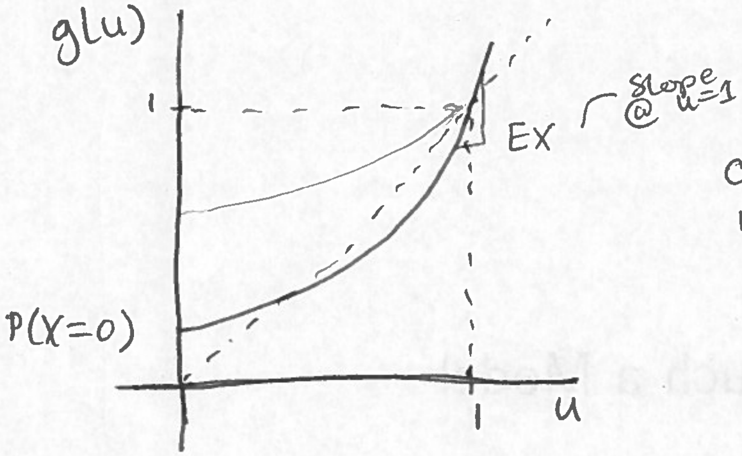
$h'(x) = -\frac{1}{x}$

$h''(x) = \frac{1}{x^2} \Rightarrow$ convex (for $x > 0$)

$$E[-\log(x)] \geq -\log EX$$

$$\Rightarrow E[\log x] \leq \log EX$$

Generating Functions – Graphical Representation of useful Facts



cross 45° line at most 1x

X_0 number of cases @ time 0 $=1$

$g_0 = E[u^{X_0}] = u^0 P(X \cancel{=} 0) + u P(X=1) + u^2 \cancel{P}(X=2)$

$g_0(u) = u$

X_1 number of cases in 2nd gen, X_2, X_3, X_4, ..., etc.

Assumption – every case of disease produces a random number of cases. Independent, identical, unchanging (over time)

Has generating function g

Mean number of cases that a case can cause (everyone susceptible) is called R

$g'(1) \equiv R$

X_1 has gen fxn $g_1(u) = g(u)$

X_2 has gen fxn

$$E[u^{X_2}] = E[E[u^{X_2} | X_1]] \quad \text{Adam's Law}$$

$$= E[u^{X_2} | X_1 = 0] P(X_1 = 0) + E[u^{X_2} | X_1 = 1] P(X_1 = 1) + E[u^{X_2} | X_1 = 2] P(X_1 = 2) + \ldots \text{ etc.}$$

$E[u^{X_2} | X_1 = 0] = 1$

$E[u^{X_2} | X_1 = 1] = g(u)$

$E[u^{X_2} | X_1 = 2]$

Person 1 } need gf of
Person 2 } the sum

Gen fxn of a sum of 2 indep RVs

$E[u^{W+V}] = E[u^W u^V] = E[u^W] E[u^V]$

If W & V i.i.D. $\Rightarrow E[u^{W+V}] = (g(u))^2$

$\Rightarrow$

$$E[u^{X_2}] = \cancel{g}\sum_{i=0}^{\infty} (g(u))^i P(X_1 = i)$$

$$= g(g(u))$$

$X_3 \quad g(g(g(u)))$

$X_n \quad g^{(n)}(u)$

$X_{n+1} \quad g(g_n(u))$

$$g'_{n+1}(u) = g'(g_n(u))\, g_n'(u)$$

$u \to 1$

$$E[X_{n+1}] = g'(1)\, g_n'(1)$$

$$= R\, E[X_n] \quad \text{geometric!}$$

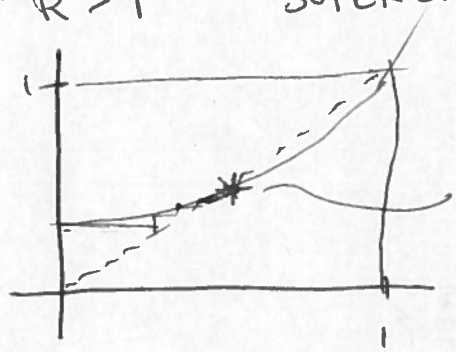
$X_n = 0$? Probability goes extinct by nth gen

$g^{(n)}(0)$?

Extinct eventually $\lim_{n \to \infty} g^{(n)}(0)$

$g(0)$	extinct @ time 0
$g(g(0))$	extinct @ time 1
$g(g(g(0)))$	etc.
$g(g(g(g(0))))$	

IF $R > 1$ SUPERCRITICAL

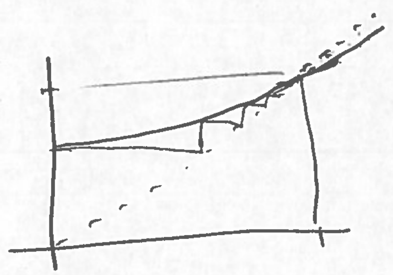


extinction probability

Extinction: probability at pt where gf crosses 45° line

probability < 1

IF $R < 1$ SUBCRITICAL



Epidemic goes extinct w/ probability 1

Epidemic "almost surely" goes extinct

Threshold theorem: $R < 1 \Rightarrow$ epidemic goes extinct w/ prob. 1

IF $R = 1$ CRITICAL

45° line is a tangent at $u = 1$

probability of extinction is still 1!

even though $E[g^{(n)}(0)] = 1$ avg of 1 case @ all times

Measles $\quad R_{eff} \approx \underbrace{(0.1)}_{\text{susceptible fraction}}\underbrace{(7)}_{R_0} = 0.7$

$$c = \text{cluster size} = \sum_n \mu_0 R^n = \frac{\mu_0}{1-R} \qquad \mu_0 \equiv 1$$

$$\Rightarrow \cancel{R<1} \quad c = \frac{1}{1-R}$$

40 Disneyland

$\downarrow$

130 total

$$c_{avg} = \frac{130}{40} \qquad R = \frac{c-1}{c} = \frac{\frac{130}{40} - 1}{130/40}$$

$$= \frac{90}{130} = 0.7$$

Parametric Bootstrap

We saw $R = 0.7$

If $R = 0.9$ what is prob. observe 0.7 ?

If $R = 0.4$ " " " " ?

Determine:

R_{upper} & R_{lower} such that R_{obs} is at the 2.5^{th} percentile.

Confidence Interval ~~R_{upper}~~

(R_{lower}, R_{upper})

Estimate: $R_{observed}$