Lecture 7

2/19/15

$\boxed{\text{Final size formula}}$

$$X \rightarrow Y \rightarrow Z$$

susceptible infectious removed

$$\frac{dx}{dt} = -\beta^* x y$$

$$\frac{dy}{dt} = \beta^* x y - \rho y$$

$$\frac{dz}{dt} = \rho y$$

$$\underbrace{1 - \underbrace{e^{-R_0 z}}_{\text{poisson distributed hits} \rightarrow \text{prob of no hits}}}_{\#\text{ w/ at least one hit}} = \underbrace{z}_{\text{number infected @ end of epidemic}}$$

$$y = -\frac{1}{\beta^* x}\frac{dx}{dt}$$

$$dz = \rho y \, dt$$

$$dz = \rho\left(-\frac{1}{x\beta^*}\right)\frac{dx}{dt}\, dt$$

$$\int dz = \int -\frac{\rho}{x\beta^*} \cancel{dt}\, dx$$

$$z(T) - z(0) = -\frac{\rho}{\beta^*} \ln \frac{X(T)}{X(0)}$$

$$z(\infty) = -\frac{\rho}{\beta^*} \ln x(\infty)$$

$$-\frac{\beta^*}{\rho} z(\infty) = \ln x(\infty)$$

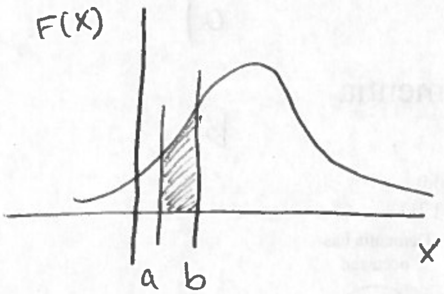
$$e^{-\frac{\beta^*}{\rho} z(\infty)} = 1 - z(\infty)$$

$\Rightarrow$ final size formula

Final Size Formula

Number of ppl who got infected is the number of ppl who could have infected that number

Continuous probability



$$P(X \in (a,b)) = \int_a^b F(x)\,dx$$

F(X) probability density

$$P(X \in [a,a]) = 0 \Rightarrow P(X=a) = 0$$

Probability 0 does not mean impossible

The real numbers are uncountable

$\boxed{\text{Discrete}}$

$$EX = \sum_i x_i P(X=x_i)$$

$\boxed{\text{Continuous}}$

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx$$

Normal – Standard Normal

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Waiting Time Variables T $\quad P(T<0) = 0$

No negative waiting times

Hazard

$$\frac{P(T \in (t, t+\Delta t])}{P(T>t)\,\Delta t}$$

$$\boxed{\lambda = \lim_{\Delta t \to 0} \frac{P(T \in (t+\Delta t])}{P(T>t)\,\Delta t}}$$

$$\int_{-\infty}^{t} f(t)\,dt = P(T \le t) = F(t)$$

Cumulative Density Function

$F(-\infty) = 0$ - Go far enough back & the event hasn't happened

$F(\infty) = 1$ - Go forward enough and the event has happened

Monotone non-decreasing = F cannot decrease as t increases!
F can be flat

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{f(t)\,\Delta t}{(1-F(t))\,\Delta t}$$

$$\lambda(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{S(t)} \qquad \frac{\text{density}}{\text{survival fxn}}$$

where $S(t) = 1 - F(t)$

<u>Example:</u> Suppose I want a density fxn w/ constant hazard

Facts:

$$\frac{dF}{dt} = f$$

$$1 - F(t) = S(t) \Rightarrow -\frac{dF}{dt} = \frac{dS}{dt}$$

$$\lambda = \frac{f(t)}{S(t)}$$

$$-\int_0^{t'} \lambda\,dt = \int_0^{t'} \frac{dS}{S}$$

$$e^{-\lambda t'} = S(t')$$

$$S(t)\,\lambda = f(t)$$

$$S(t)\,\lambda = -\frac{dS}{dt}$$

$$\lambda = -\frac{1}{S}\frac{dS}{dt}$$

$$-\lambda\,dt = \frac{dS}{S}$$

Constant Hazard, Exponential Survival!

$$S(t) = e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$\frac{dF}{dt} = -\frac{d}{dt} e^{-\lambda t} = -(-\lambda) e^{-\lambda t} = \lambda e^{-\lambda t}$$

$$\boxed{f(t) = \lambda e^{-\lambda t}}$$

$$ET = \int_{-\infty}^{\infty} t f(t)\, dt = \int_0^{\infty} t \lambda e^{-\lambda t}\, dt$$

$$u = t \qquad dV = e^{-\lambda t}\, dt$$

$$du = dt \qquad V = -\frac{1}{\lambda} e^{-\lambda t}$$

$$ET = \lambda \left\{ -\frac{1}{\lambda} e^{-\lambda t}\, t \Big|_0^{\infty} - \int_0^{\infty} \left(-\frac{1}{\lambda} e^{-\lambda t}\right) dt \right\}$$

$$= \cancel{\lambda} \int_0^{\infty} e^{-\lambda t}\, dt = -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda}$$

Expected waiting time is the reciprocal of the rate for an exponential survival function. (or constant hazard)

Example:

t^* - survive to this - what is survival time after that?

$$P(T - t^* > u \mid T > t^*)$$

$$= \frac{P(T - t^* > u \cap T > t^*)}{P(T > t^*)}$$

$$= \frac{P(T - t^* > u)}{P(T > t^*)} = \frac{P(T > u + t^*)}{P(T > t^*)}$$

$$= \frac{e^{-\lambda(u+t^*)}}{e^{-\lambda t^*}}$$

$$= e^{-\lambda u}$$

If you wait t^*, the expected amount you have to wait is the same it was at the begining.

<u>Memorylessness</u>

<u>K-M model</u> - Analyze very start of epidemic to get $\underline{R_0}$

$Y \xrightarrow{\rho}$ Drop a bunch of infectives into Y and watch them recover

$$\frac{dY}{dt} = -\rho Y \qquad Y = Y(0)e^{-\rho t}$$

Mean duration of infectivity: $\frac{1}{\rho}$

$\frac{1}{\rho}$ avg time infected

KM Model Incidence rate

$$\frac{dx}{dt} = -\beta X Y$$

$$= -X\underbrace{(\beta Y)}_{\text{hazard of removal}}$$

At beginning

$$X \cong N$$

$$\frac{dI}{dt} = \beta N Y$$

Cumulative infections @ start

$$\frac{dI}{dt} = \beta N Y^{(0)}$$

$$\int_{t=0}^{t=t'} dI = \int_0^{t'} \beta N Y^{(c)} \, dt$$

$$I(t') - \cancel{I(0)}^{\,0} = \beta N \int_0^{t'} e^{-\rho t} Y_0(0) \, dt$$

$$I(t) = \beta N Y_0(0) \int_0^{t^*} e^{-\rho t'} \, dt'$$

$$\underline{I(\infty) = \frac{\beta N}{\rho} Y(0)}$$

Total # new infectives → per initial case

$$\boxed{\frac{\beta N}{\rho} = R_0}$$

Example:

Suppose 2 waiting times T_1, T_2

T_1: death disease 1 $\quad$ T_2: death disease 2

$$T = \min(T_1, T_2) = T_1 \wedge T_2$$ Time of death?

T_1 T_2 independent & exponential

$$T > t^* = T_1 > t^* \cap T_2 > t^*$$

$$P(T > t^*) = P(T_1 > t^*) P(T_2 > t^*) = e^{-\lambda_1 t^*} e^{-\lambda_2 t^*}$$

$$P(T > t^*) = e^{-(\lambda_1 + \lambda_2) t^*}$$

Hazard $\lambda_1 + \lambda_2$

2 indep things → <u>just add hazards</u>!

Competing Exponential Risks

$$\begin{tikzcd} & X \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

Independent processes

* You cannot add hazards unless you assume independence *

$$\frac{dX}{dt} = -(\lambda_1 + \lambda_2) X = -\lambda_1 X - \lambda_2 X$$

$$\frac{dM_1}{dt} = \lambda_1 X$$

$$\frac{dM_2}{dt} = \lambda_2 X$$

$$X(t) = e^{-(\lambda_1 + \lambda_2) t} X(0)$$

Aside: Assume only λ_1 (no λ_2)

Expected deaths in Δt = number @ risk x risk

$$\Delta X = -X \lambda_1(t) \Delta t$$

$$\frac{\Delta X}{\Delta t} = -\lambda_1 X$$

Adding the hazards $\Rightarrow$ independence

$$M_1(t) = \lambda_1 \int_0^t X(0) e^{-(\lambda_1 + \lambda_2) t} \, dt$$

$$M_1(\infty) = \frac{\lambda_1}{\lambda_1 + \lambda_2} X(0)$$

Fraction in M_1 is $\frac{\lambda_1}{\lambda_1 + \lambda_2}$

other models in Epidemiology

$$X \rightleftarrows X^*$$ Loops

$$Y_1 \to Y_2 \to Y_3 \to Y_4 \ldots$$ Chains