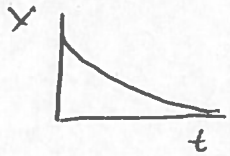
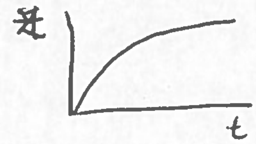
3/4/2016

Lecture 9

$X \xrightarrow{\lambda} Z$





Z corresponds to

$P(T<t)$

Exponential Distribution

- memoryless – addition waiting time does not depend on how long already waited
- constant hazard – hazard = probability per unit time given that the event hasn't already happened

$X \xrightarrow{\lambda(t)}$

$$\frac{dX}{dt} = -\lambda X$$

$$\frac{dX}{X} = -\lambda(t)\,dt$$

$$\int_{t=0}^{t=t'} \frac{dX}{X} = -\int_{t=0}^{t=t'} \lambda(t)\,dt$$

HW Problem – tiny interval vs. longer one

$$\ln X(t') - \ln X(0) = -\int_{t=0}^{t=t'} \lambda(t)\,dt$$

$$\frac{X(t')}{X(0)} = e^{-\int_0^{t'} \lambda(t)\,dt}$$

CONSTANT HAZARD – EXPONENTIAL WAITING TIME!

HIV

MSM in SF – 1981 HIV prev = 1%, 1983 HIV prev = 50%

But since HIV → AIDS is not exponential!
(Incidence of AIDS was not highest in 1983)

Distributed Delays

$$\boxed{X_1} \xrightarrow[\alpha_1]{} \boxed{X_2} \xrightarrow[\alpha_2]{} \boxed{X_3}$$

* More realistic waiting time distributions can be approximated with chains of exponentials

$$\bar{T} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$$

$$X_1 \xrightarrow{\lambda} X_2 \xrightarrow{\lambda} Z$$

$X_1(0) = 1 \qquad X_2(0) = 0 \qquad Z(0) = 0$

What is the distribution of ~~[illegible]~~ waiting times? That gives us the proportion in Z over time

$$\frac{dX_1}{dt} = \dot{X_1} = -\lambda X_1$$

Z is the cumulative distribution fxn

$$\frac{dX_2}{dt} = \lambda X_1 - \lambda X_2$$

what is the distribution of the sum of two independent identical exponential waiting times

$$\frac{dZ}{dt} = \lambda X_2$$

	Continuous		Discrete
1	Exponential	$\longleftrightarrow$	~~Negative Binomial~~ Geometric
Sum of 2	??	$\longleftrightarrow$	Negative Binomial

$$X_1 = e^{-\lambda t}$$

$$\frac{dX_2}{dt} + \lambda X_2 = \lambda e^{-\lambda t}$$

$$\frac{dx_2}{dt} e^{\lambda t} + \lambda x_2 e^{\lambda t} = \lambda$$

$$\frac{d}{dt}\left(x_2 e^{\lambda t}\right) = \lambda \qquad \int_0^{t'} d\left(x_2 e^{\lambda t}\right) = \int_0^{t'} \lambda \, dt$$

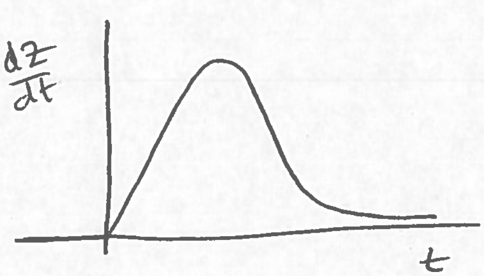
$$x_2(t') e^{\lambda t'} - x_2(0) e^{0} = \lambda t'$$

$$x_2(t) = \lambda t e^{-\lambda t}$$

$$\frac{dz}{dt} = \lambda x_2 = \lambda^2 t e^{-\lambda t}$$

z is the CDF

so $\frac{dz}{dt}$ gives the density fcn of the arrival time



Gamma distribution withe shape parameter 2

Add n indep exponential, I get a gamma w/ shape parameter n.

χ^2 distribution is also a gamma distribution

$$\underbrace{S \to E \to I}_{\text{more realistic waiting time}} \to R$$

$$S \to E_1 \to E_2 \to I \to R$$

$$S \to E \to I_1 \to I_2 \to R$$

$$S \to E_1 \to E_2 \to I_1 \to I_2 \to R \qquad \underline{\text{etc.}}$$

R_0 for chains and forks

$$X \to \overset{\beta_1}{I_1} \xrightarrow{\gamma_1} \overset{k\beta_1}{I_2} \xrightarrow{\gamma_2} R$$

What is R_0?

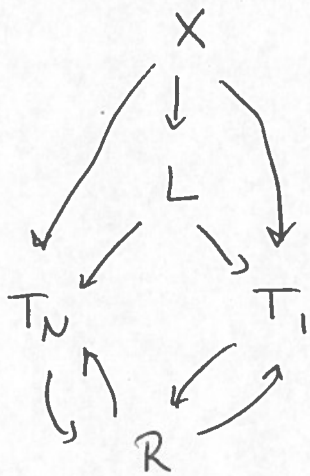
$$\frac{\beta_1}{\gamma_1} + \frac{k\beta_2}{\gamma_2}$$

FIV

$$\begin{tikzcd}
 & & \overset{\beta_1}{I_1} \arrow[r, "\rho_1"] & \overset{\beta_2}{I_2} \arrow[r, "\rho_2"] & R & \text{Symptomatic} \\
X \arrow[r] & E \arrow[ur, "\gamma \quad p"] \arrow[dr, "1-p"'] & & & & \\
 & & \underset{\beta_1'}{I_1'} \arrow[uurr, "\rho_1'"'] & & & \text{asymptomatic}
\end{tikzcd}$$

$$R_0 = p\left(\frac{\beta_1}{\rho_1} + \frac{\beta_2}{\rho_2}\right) + (1-p)\left(\frac{1}{\rho_1'}\,\beta_1'\right)$$

TB Model, Blower 1995



Immigration – Death

$$\xrightarrow{\lambda} \; X \; \xrightarrow{\mu}$$

$$\dot{X} = \lambda - \mu X$$

$$\bar{X} = \lambda/\mu = \lambda \frac{1}{\mu}$$

Prevalence = incidence · duration

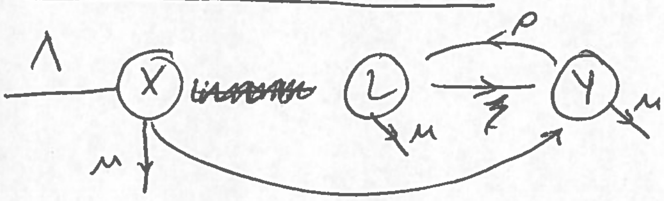
Most infected & remain latent

Small frxn become cases
- some infectious & non-infectious

Spontaneous remission

Modified SI process

HSV-2 Model – Loops



$$\dot{X} = \lambda - \mu X - \beta X \frac{Y}{N}$$

$$\dot{Y} = \beta X \frac{Y}{N} - \rho Y - \mu Y + \xi L$$

$$\dot{L} = -\mu L + \rho Y - \xi L$$

$$R_0 = \underbrace{\frac{1}{\rho+\mu}\beta}_{\text{cases per episode}} + k \frac{\beta}{\rho+\mu} + k^2 \frac{\beta}{\rho+\mu} + \ldots$$

where k is the probability of each episode

$$\Rightarrow R_0 = \frac{\beta}{\rho+\mu} \frac{1}{1-K} \quad \text{using sum of a geometric}$$

$$K = \underbrace{\frac{\rho}{\rho+\mu}}_{\substack{\text{prob} \\ \text{you recover} \\ \text{(and not die)}}} \cdot \underbrace{\frac{\xi}{\xi+\mu}}_{\substack{\text{prob} \\ \text{you relapse} \\ \text{(and not die)}}}$$

Modeling Ageing

$$(A_1) \rightarrow (A_2) \rightarrow (A_3) \rightarrow (A_4) \rightarrow$$

compartmental ageing

Alternate approach:

$X(a,t)$ density of people in age at time t

$X(a,t)\Delta a$ # of ppl w/ age btwn a and $a+\Delta a$

Follow a cohort!

$$X(a+\Delta a, t+\Delta a) = X(a,t)(1-\mu(a)\Delta a)$$

$$X(a+\Delta a, t+\Delta a) - X(a,t) = -X(a,t)\mu(a)\Delta a$$

$$X(a+\Delta a, t+\Delta a) - X(a, t+\Delta a) + X(a, t+\Delta a) - X(a,t) = -X(a,t)\mu(a)\Delta a$$

$$\frac{X(a+\Delta a, t+\Delta a) - X(a, t+\Delta a)}{\Delta a} + \frac{X(a, \overset{t+\Delta t}{t+\Delta a}) - X(a, t)}{\underset{\Delta t}{\Delta a}} = -X\mu$$

Take limit as $\Delta a \to 0$, $\Delta t \to 0$

$$\frac{\partial X}{\partial a} + \frac{\partial X}{\partial t} = -\mu(a) X \qquad \text{PDE Age Structure Model}$$

McKendrick-von Foerster Equation