Lecture 10

3/18/2016

$$\begin{cases} N_1(\tau+1) = a_{11} N_1(\tau) + a_{12} N_2(\tau) \\ N_2(\tau+1) = a_{21} N_1(\tau) + a_{22} N_2(t) \end{cases} \qquad (a_{ij} \geq 0)$$

$$\begin{bmatrix} N_1(\tau+1) \\ N_2(\tau+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} N_1(\tau) \\ N_2(\tau) \end{bmatrix}$$

$$\mathbb{N}(\tau+1) = \mathbb{A}\,\mathbb{N}(\tau)$$

Homework Problem 1

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 6 & -1 \\ 1 & 2 & 1 & 1 \end{bmatrix} = ?$$

Since $\mathbb{N}(\tau+1) = \mathbb{A}\,\mathbb{N}(\tau)$

$$\Rightarrow \mathbb{N}(\tau) = \mathbb{A}^{\tau}\,\mathbb{N}(0)$$

$$\mathbb{A}^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{bmatrix}$$

what if $a_{12} = a_{21} = 0$?

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}^{\tau} = \begin{bmatrix} a_{11}^{\tau} & 0 \\ 0 & a_{22}^{\tau} \end{bmatrix}$$

$$\begin{bmatrix} N_1(\tau) \\ N_2(\tau) \end{bmatrix} = \begin{bmatrix} a_{11}^{\tau} & 0 \\ 0 & a_{22}^{\tau} \end{bmatrix} \begin{bmatrix} N_1(0) \\ N_2(0) \end{bmatrix} = \begin{bmatrix} a_{11}^{\tau} N_1(0) \\ a_{22}^{\tau} N_1(0) \end{bmatrix}$$

<u>OR</u> $\quad N_i(\tau) = a_{11}^{\tau} N_i(0) \quad$ for $i = 1,2$ <u>GEOMETRIC</u>

$0 \leq a_{ii} < 1$ geometric decay

$a_{ii} > 1$ geometric growth

If you have a positive matrix, largest eigenvalue is $\mathbb{R}$ and positive.

If disease transmission, then the largest eigenvalue corresponds to R_0.

To get eigen values:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

(generally, det $A - \lambda \mathbb{I} = 0$)

[$\mathbb{I}$ is the identity matrix

$\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A\mathbb{I} = A$]

$\mathbb{I}A = A$

Determinant - specified w/ vertical bars

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det A = ad - bc = 0$

then $ad = bc$ or $\frac{ad}{bc} = 1$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

$$\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0$$

$$\lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 + 4(a_{21}a_{12} - a_{11}a_{22})}}{2}$$

give us 2 eigen values

Look under the square root -

$$a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 + 4a_{21}a_{12}$$

$(a_{11} - a_{22})^2 + 4a_{21}a_{12}$ This has to be ≥ 0, so eigenvalues must be $\mathbb{R}$!

(No complex eigenvalues for a positive matrix)

Suppose the eigenvalues are λ_+ and λ_-

$$\begin{bmatrix} a_{11}-\lambda_+ & a_{12} \\ a_{21} & a_{22}-\lambda_+ \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \cancel{\lambda_+ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbb{A}\,\mathbb{v} = \lambda_+ \mathbb{I}\,\mathbb{v}$$

$$\mathbb{A} - \lambda_+ \mathbb{I}\,\mathbb{v} = 0$$

$$\mathbb{B} = \mathbb{A} - \lambda_+ \mathbb{I} \longrightarrow \mathbb{B}\,\mathbb{v} = 0$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + v_2 \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Home work Problem 2

Convince yourself that any matrix times any vector is a linear combination of the columns of ~~an~~ that matrix.

~~e.g. $\mathbb{A}\,\mathbb{v} = v_1 \mathbb{A} + v_2 \mathbb{A}$~~

Solving these eqns, we get

$$\mathbb{A}\,\mathbb{v}_+ = \lambda_+ \mathbb{v}_+$$

$$\mathbb{A}\,\mathbb{v}_- = \lambda_- \mathbb{v}_-$$

$\mathbb{v}_+$ and $\mathbb{v}_-$ eigenvectors

Homework problem 3

Example

$$A = \begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$$

what are the eigenvalues and eigenvectors of A?

$$\begin{cases} 1.12 \\ -0.12 \end{cases} \text{ Eigenvalues}$$

$$\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.2 \end{bmatrix} \begin{bmatrix} 0.88 \\ 0.48 \end{bmatrix} = \begin{bmatrix} (0.88)(0.8) + (0.48)(0.6) \\ (0.88)0.5 + (0.48)(0.2) \end{bmatrix} =$$

$$\approx 1.12 \begin{bmatrix} 0.88 \\ 0.48 \end{bmatrix}$$

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Now do this for $A = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.5 \end{bmatrix}$

what are the eigenvectors and values of A?
(you are welcome to check w/ R or Sage, but do by hand)

$$\mathbb{N}(0) = k_1 \mathbb{V}_+ + k_2 \mathbb{V}_-$$

$$A\,\mathbb{N}(0) = A(k_1 \mathbb{V}_+ + k_2 \mathbb{V}_-)$$

$$= k_1 \lambda_+ \mathbb{V}_+ + k_2 \lambda_- \mathbb{V}_-$$

$$A\,\mathbb{N}(0) = k_1 \lambda_+^2 \mathbb{V}_+ + k^2 \lambda_-^2 \mathbb{V}_-$$

$$A^\tau\,\mathbb{N}(0) = k_1 \lambda_+^\tau \mathbb{V}_+ + k_2 \lambda_-^\tau \mathbb{V}_-$$

If

Disease dies out $\begin{cases} 0 \le |\lambda_+| < 1 \\ 0 \le |\lambda_-| < 1 \end{cases}$

epidemic $\begin{cases} |\lambda_+| > 1 \\ |\lambda_-| > 1 \end{cases}$

$\max |\lambda_\pm| < 1$

$\Rightarrow$ No epidemic

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