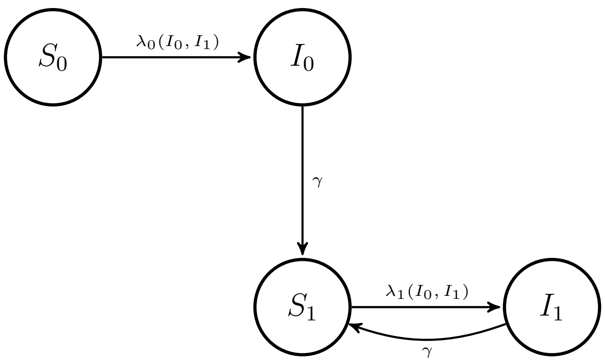
Homework 8

NB: Flow rates on compartmental model diagrams are per capita flow rates unless otherwise noted.

Exercise 1: Immunity to Reinfection 4pts

Suppose we want to model a disease where reinfection is possible, but individuals who have been previously infected have partial protection against reinfection. The compartmental model diagram below depicts such a model. Using this diagram, determine a $\lambda_0(I_0, I_1)$ and a $\lambda_1(I_0, I_1)$ that accomplishes partial immunity to reinfection and write the four differential equations corresponding to this model. Answers may vary. $\blacksquare$



$$\dot{S}_0 = -\lambda_0 S_0$$

$$\dot{I}_0 = \lambda_0 S_0 - \gamma I_0$$

$$\dot{S}_1 = -\lambda_1 S_1 + \gamma I_0 + \gamma I_1$$

$$\dot{I}_0 = \lambda_1 S_1 - \gamma I_1$$

Where

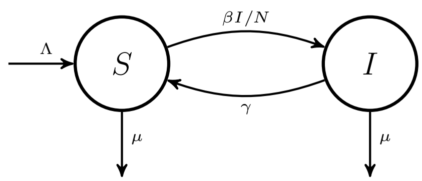
$$\lambda_1 = \frac{\zeta\beta(I_0 + I_1)}{N}$$

where $0 < \zeta < 1$ (degree of partial immunity).

$$\lambda_0 = \frac{\beta(I_0 + I_1)}{N}$$

Exercise 2: SIS Model with Death 3 pts

Write the differential equations and the R_0 for the model depicted below. [Λ is a constant inflow. You are not required to use the next generation method. You can write down R_0 and explain your reasoning.] ∎



$$\dot{S} = \Lambda - \mu S + \gamma I - \beta SI/N$$

$$\dot{I} = -\mu I - \gamma I + \beta SI/N$$

$$R_0 = \frac{\beta}{\gamma + \mu}$$

Without mortality the R_0 we got in class was β/γ. The numerator should remain unchanged because the force of infection in this model is unchanged. However, the duration of infection is now $1/(\gamma + \mu)$ and not $1/\gamma$, so the denominator needs to reflect this.

Exercise 3: SIS Model with Heterogeneity 6 pts

The model depicted above represents a heterogenous-risk SIS model (e.g. STD model with a core group and a non-core group), where

$$\lambda_0 = c_0(p_{00}\frac{I_0}{N_0} + p_{01}\frac{I_1}{N_1})\beta$$

$$\lambda_1 = c_1(p_{10}\frac{I_0}{N_0} + p_{11}\frac{I_1}{N_1})\beta.$$

The model has the following constraints:

$$p_{00} + p_{01} = 1$$

$$p_{10} + p_{11} = 1$$

$$N_1 c_1 p_{10} = N_0 c_0 p_{01}$$

Suppose we parameterize p_{00} with a parameter $q \in [0, 1]$ as follows:

$$p_{00} = \frac{c_0 N_0 + c_1 N_1 q}{c_0 N_0 + c_1 N_1}$$

(a) Find p_{01}, p_{10}, and p_{11} in terms of c_0, c_1, N_0, N_1, and q.

$$p_{01} = 1 - p_{00} = 1 - \frac{c_0 N_0 + c_1 N_1 q}{c_0 N_0 + c_1 N_1} = \frac{c_1 N_1 (1-q)}{c_0 N_0 + c_1 N_1}$$

$$p_{10} = \frac{N_0 c_0}{N_1 c_1} p_{01} = \frac{N_0 c_0}{N_1 c_1}\frac{c_1 N_1 (1-q)}{c_0 N_0 + c_1 N_1} = \frac{N_0 c_0 (1-q)}{c_0 N_0 + c_1 N_1}$$

$$p_{11} = 1 - p_{10} = 1 - \frac{N_0 c_0 (1-q)}{c_0 N_0 + c_1 N_1} = \frac{c_1 N_1 + c_0 N_0 q}{c_0 N_0 + c_1 N_1}$$

(b) What is the parameter q in public health terms? (Hint: $q = 0$ corresponds to random mixing.)

q gives the degree of assortative mixing. $q = 0$ corresponds to random mixing. $q = 1$ corresponds to completely assortative mixing. Intermediate values of q correspond to mixing that is not fully random and not fully assortative.

(c) How many cases in group 0 can one group 0 infection cause in a fully susceptible population?

$$\frac{c_0 p_{00} \beta}{\gamma}$$

(d) How many cases in group 1 can one group 1 infection cause in a fully susceptible population?

$$\frac{c_1 p_{11} \beta}{\gamma}$$

(e) How many cases in group 0 can one group 1 infection cause in a fully susceptible population?

$$\frac{c_1 p_{10} \beta}{\gamma}$$

(f) How many cases in group 1 can one group 0 infection cause in a fully susceptible population?

$$\frac{c_0 p_{01} \beta}{\gamma}$$

$\blacksquare$

Exercise 4 1 pt

How long did this assignment take you? $\blacksquare$