Lecture 6 2/12/16

Calculus

$$\frac{d}{dx} x^n = n x^{n-1} \qquad \frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Binomial

$$P(X=x) = \binom{N}{x} p^{x} (1-p)^{N-x}$$

$$EX = Np$$

$N \to \infty$, $p \to 0$ but left $\lambda = Np$ fixed

Poisson

$$P(X=i) = \frac{e^{\lambda} \lambda^{i}}{i!}$$

Discrete vs Continuous time models - both are valid & useful

Stochastic vs Deterministic models - use deterministic models to understand stochastic models

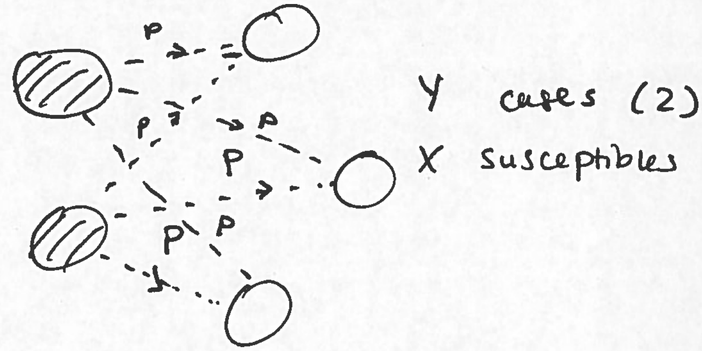
Galton - Watson - discrete & stochastic
↳ generation time

- Assortive mixing - no network structure
- No depletion of susceptibles
- No intervention

New Model (Reed-Frost Process)

X_τ susceptible

Y_τ infected



risk of infection:

$$r_\tau = 1 - (1-p)^{Y_\tau}$$

only one way to not get infected — not get infected by everyone

$$\left.\begin{aligned} Y_{\tau+1} &= \text{Binom}(X_\tau, r_\tau) \\ X_{\tau+1} &= X_\tau - Y_{\tau+1} \end{aligned}\right\} \text{ assumes recovery is 1 time step}$$

[Greenwood Process - r for $Y \geq 1$ constant]

This process terminates in finite time

$R = p(N-1)$ Effective reproduction number

Get rid of assumption that recovery is 1 time step

$$\underset{\text{susceptible}}{\boxed{X}} \longrightarrow \underset{\text{infected}}{\boxed{Y}} \longrightarrow \underset{\text{recovery}}{\boxed{Z}}$$

Recovery — γ probability of recovery per day

recoverys Binom(Y_τ, γ)

infections Binom(X_τ, r_τ)

$$Y_{\tau+1} = Y_\tau - \text{recoveries} + \text{infections}$$

$$Z_{\tau+1} = Z_\tau + \text{recoveries}$$

$$X_{\tau+1} = X_\tau - \text{infections}$$

Geometric waiting time in Y box

Make time step Δt instead of 1 day

$X_{\tau+\Delta\tau}$ etc.

$\gamma = \rho\Delta t$ $\qquad$ ρ is the <u>hazard</u>

$p = \beta\Delta t$

$$r_\tau = 1 - (1-\beta\Delta\tau)^{Y_\tau}$$

$$\sum_{i=0}^{Y_\tau} \binom{Y_\tau}{i} (-\beta\Delta t)^{Y_\tau - i} = (1-\beta\Delta\tau)^{Y_\tau}$$

$$= 1 + (-\beta\Delta\tau)Y_\tau + O(\Delta\tau)$$

$$\Rightarrow r_\tau \approx 1 - (1 - \beta Y_\tau \Delta\tau) = \beta Y_\tau \Delta\tau$$

Expected recoveries $= Y_\tau \rho \Delta\tau$

Expected infectives $= X_\tau \beta Y_\tau \Delta\tau$

Replace RVs w/ avgs over short time periods & make new model!

$$\left\{\begin{aligned} X_{\tau+\Delta\tau} &= X_\tau - \beta X_\tau Y_\tau \Delta\tau \\ Y_{\tau+\Delta\tau} &= Y_\tau - Y_\tau\rho\Delta\tau + \beta X_\tau Y_\tau \Delta\tau \\ Z_{\tau+\Delta\tau} &= Z_\tau + Y_\tau \rho\Delta\tau \end{aligned}\right. \tag{2}$$

Discrete time & Deterministic

- small time limit
- large ppl limit

(3)

$$\begin{cases} \dfrac{X_{\tau+\Delta\tau} - X_\tau}{\Delta\tau} = -\beta X_\tau Y_\tau \\[2ex] \dfrac{Y_{\tau+\Delta\tau} - Y_\tau}{\Delta\tau} = \beta X_\tau Y_\tau - \rho Y_\tau \\[2ex] \dfrac{Z_{\tau+\Delta\tau} - Z_\tau}{\Delta\tau} = \rho Y_\tau \end{cases}$$

$\lim_{\Delta\tau \to 0}$

(4)

$$\begin{cases} \dfrac{dX}{dt} = -\beta X Y \\[2ex] \dfrac{dY}{dt} = \beta X Y - \rho Y \\[2ex] \dfrac{dZ}{dt} = \rho Y \end{cases}$$

<u>Kermack McKendrick Model</u>

Deterministic model in continuous time

- McKendrick
- Kermack <u>1927</u>

<u>SIR model.</u>

Non linear ordinary differential equations

$$\frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt} = 0 \Rightarrow N \text{ constant}$$

Stochastic $\rightarrow$ Deterministic

- replace RVs w/ their expectations

$X = Nx$

$Y = Ny$

$Z = Nz$

lower case = fractions
uppercase = numbers

$$\frac{dX}{dt} = \frac{d}{dt}(Nx) = N\frac{dx}{dt}$$

$$\begin{cases} \cancel{N}\dfrac{dx}{dt} = -\beta \cancel{N} x N y \\[2ex] \cancel{N}\dfrac{dy}{dt} = \beta \cancel{N} x N y - \rho \cancel{N} y \\[2ex] \cancel{N}\dfrac{dz}{dt} = \rho \cancel{N} y \end{cases}$$

$$\beta^* = \beta N$$

$$\begin{cases} \dfrac{dx}{dt} = -\beta^* x y \\[2ex] \dfrac{dy}{dt} = \beta x y - \rho y \\[2ex] \dfrac{dz}{dt} = \rho y \end{cases}$$

Large $N \Rightarrow N-1 \simeq N$

$X \simeq N \quad Y = 1$

At Beginning of epidemic

$$\frac{dY}{dt} = (\beta N - \rho) Y = (\beta^* - \rho) Y$$

$$\beta^* - \rho > 0 \Rightarrow \frac{dY}{dt} > 0$$

Epidemic increasing if $\beta^* - \rho > 0$

decreasing if $\beta^* - \rho < 0$

Reproduction # $\dfrac{\beta^*}{\rho} = R_0$

if $R_0 > 1$ increase at start

if $R_0 < 1$ decrease at start

At start

$$\frac{dz}{dt} = \rho \frac{-1}{\beta^* x} \frac{dx}{dt} = \frac{-\rho}{\beta^* x} \frac{dx}{dt} = \frac{-1}{R_0 x} \frac{dx}{dt}$$

$$\int_0^t \frac{dz}{dt} \, dt = -\frac{1}{R_0} \int_0^t \frac{dx}{dt} \frac{1}{x} \, dt$$

$$z(t) - z(0) = -\frac{1}{R_0} \ln \frac{x(t)}{x(0)}$$

$t \to \infty$ Epidemic over!

$$z(\infty) = -\frac{1}{R_0} \ln x(\infty$$

Also $x(\infty) = 1 - z(\infty)$ b/c no infectives @ end of epidemic

$$z(\infty) = -\frac{1}{R_0} \ln (1 - z(\infty))$$

$$\Rightarrow \quad z(\infty) = 1 - e^{-R_0 z(\infty)}$$

Final attack size is the number of people who the final attack size could infect

<u>Final size formulation</u>